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LETTER TO THE EDITOR

Hopf bifurcation in a binary liquid: exact upper bound on the frequency

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Abstract. We prove that for the Hopf bifurcation in a binary liquid, the frequency ω has an exact upper bound given by $\omega_{max}=(kR/\sigma)^{1/2}$ for $k>\frac{2}{3}$, where k is the separation parameter, σ is the Prandtl number and R is the thermal Rayleigh number. The result is true for any boundary conditions. In the process, an inequality between the concentration and temperature fields is established.

The onset of convection in binary liquids has attracted a good deal of attention lately [1-5] because of the possibility of the codimension-two point, the non-equilibrium tricritical point and the travelling wave instability at the point of the Hopf bifurcation. However, no exact results are known for the onset of oscillatory convection (i.e. the Hopf bifurcation) in these mixtures. In this letter we establish a rigorous upper bound on the frequency of the oscillatory state over a certain range of the separation parameter k for any boundary condition. In particular, we prove that if ω is the frequency of the oscillation in units of ν/d^2 (where ν is the kinematic viscosity and d is the plate separation in a Rayleigh-Benard geometry), then $\omega^2<kR/\sigma$ for $k>\frac{2}{3}$. Here σ is the Prandtl number of the fluid and R is the thermal Rayleigh number.

We scale distances by d, time by d^2/ν and use the field variables w (z component of velocity scaled by λ/d, where λ is the thermal diffusivity), θ (temperature fluctuation δT from the conduction state scaled by ΔT, the temperature difference between the plates) and $\eta=\phi-\theta$ (ϕ is the concentration fluctuation δc from the conduction state scaled by Δc, the concentration difference between the plates). In terms of these variables the linear stability equations are [5] (we drop the negligible Dufour term):

$$(D^2-a^2)(D^2-a^2-p)w=Ra^2(1-k)\theta-kRa^2\eta \tag{1}$$

$$(D^2-a^2-p\sigma)\theta=-w \tag{2}$$

$$\left(D^2-a^2-p\frac{\sigma}{s}\right)\eta=-\frac{w}{s}+\frac{\sigma}{s}p\theta=\frac{1}{s}(D^2-a^2)\theta. \tag{3}$$

In the above, k is the separation parameter ($k=(k_T/T)\beta/\alpha$ where k_T is the thermodiffusivity, α is the thermal expansion coefficient and β determines the fractional change in density as the concentration of the lighter component is changed), the thermal Rayleigh number R is given by

$$R=\frac{\alpha(\Delta T)\,d^3g}{\lambda\nu} \tag{4}$$

s is the Lewis number which is the ratio of the mass diffusivity to the thermal diffusivity, p is the relaxation rate of the fluctuating fields w, θ and η, a is the dimensionless wavenumber in the xy plane and D is the operator d/dz. We note that (1)-(3) are a double-eigenvalue equation in R and p, which are to be prescribed boundary conditions.

The realistic boundary condition on the velocity field implies 'no-slip' on the boundaries which leads to $w = Dw = 0$ on the boundaries $z = 0$ and 1. If the bounding plates have high conductivity, then temperature fluctuations vanish on the plates and we have $\theta = 0$ on $z = 0$ and 1. The realistic boundary condition on the concentration field is to have the normal component of the mass current vanish at the boundaries and this implies $D\eta = 0$ on $z = 0$ and 1. Thus the realistic boundary conditions are $w = Dw = \theta = D\eta = 0$ on $z = 0$ and 1.

Under these realistic conditions, integrations by parts establishes the following results:

$$\int_0^1 w^* D^{2n} w \, dz = (-1)^n \int_0^1 |D^n w|^2 \, dz \qquad n = 1, 2 \tag{5a}$$

$$\int_0^1 \theta^* D^2 \theta \, dz = -\int_0^1 |D\theta|^2 \, dz \tag{5b}$$

$$\int_0^1 \eta^* D^2 \eta \, dz = -\int_0^1 |D\eta|^2 \, dz. \tag{5c}$$

These results are true for the idealised boundary conditions as well and hence the result obtained is independent of the nature of the boundary conditions. We now multiply (1) by w^* and integrate from 0 to 1 to obtain†

$$\begin{aligned}\int w^*(D^2 - a^2)^2 w \, dz - p \int w^*(D^2 - a^2) w \, dz &= Ra^2(1-k) \int w^* \theta \, dz - kRa^2 \int w^* \eta \, dz \\ &= -Ra^2(1-k) \int \theta(D^2 - a^2 - \sigma p^*)\theta^* \, dz - kRa^2 \int w^* \eta \, dz \end{aligned} \tag{6}$$

where (2) has been used. From (3) we note that

$$\int \eta(D^2 - a^2)\eta^* \, dz - p^* \frac{\sigma}{s} \int |\eta|^2 \, dz = -\frac{1}{s} \int w^* \eta \, dz + \frac{\sigma}{s} p^* \int \theta^* \eta \, dz$$

or

$$\int w^* \eta \, dz = \sigma p^* \int \theta^* \eta \, dz + \sigma p^* \int |\eta|^2 \, dz - s \int \eta(D^2 - a^2)\eta^* \, dz. \tag{7}$$

Inserting in (6), we arrive at

$$\begin{aligned}\int w^*(D^2 - a^2)^2 w \, dz - p \int w^*(D^2 - a^2) w \, dz &= Ra^2(1-k) \int \theta(D^2 - a^2 - \sigma p^*)\theta^* \, dz - kRa^2 \sigma p^* \int \theta^* \eta \, dz \\ &\quad - kRa^2 \sigma p^* \int |\eta|^2 \, dz + kRa^2 s \int \eta(D^2 - a^2)\eta^* \, dz. \end{aligned} \tag{8}$$

Using (5a)-(5c) and equating the imaginary parts of either side of the above equation,

† Our proof uses manipulations similar to those of Pellew and Southwell [6].

we find at the onset point (i.e. $p_1 = 0$, where $p = p_1 + \mathrm{i}p_2$, with $p_2 \neq 0$) of the oscillatory instability that

$$\int |Dw|^2\,\mathrm{d}z + a^2 \int |w|^2\,\mathrm{d}z + Ra^2\sigma(1-k) \int |\theta|^2\,\mathrm{d}z - kRa^2\sigma \int |\eta|^2\,\mathrm{d}z$$
$$= kRa^2\sigma\,\mathrm{Re} \int \theta^*\eta\,\mathrm{d}z \leqslant \tfrac{1}{2}kRa^2\sigma \int (|\theta|^2 + |\eta|^2)\,\mathrm{d}z. \tag{9}$$

Thus,

$$\int |Dw|^2\,\mathrm{d}z + a^2 \int |w|^2\,\mathrm{d}z + \sigma Ra^2(1-\tfrac{3}{2}k) \int |\theta|^2\,\mathrm{d}z - \tfrac{3}{2}kRa^2\sigma \int |\eta|^2\,\mathrm{d}z \leqslant 0$$

leading to

$$\int |\eta|^2\,\mathrm{d}z \geqslant \tfrac{2}{3}k^{-1}(1-\tfrac{3}{2}k) \int |\theta|^2\,\mathrm{d}z. \tag{10}$$

We now return to (1), multiply by w^* once again and integrate from 0 to 1, using

$$\eta = \frac{s}{\sigma p}(D^2 - a^2)\eta + \frac{w}{\sigma p} - \theta \tag{11}$$

from (3) to find

$$\int w^*(D^2-a^2)^2 w\,\mathrm{d}z - p \int w^*(D^2-a^2)w\,\mathrm{d}z$$
$$= Ra^2(1-k) \int w^*\theta\,\mathrm{d}z - kRa^2 \int \left(\frac{s}{\sigma p}(D^2-a^2)\eta + \frac{w}{\sigma p} - \theta\right) w^*\,\mathrm{d}z$$
$$= Ra^2 \int w^*\theta\,\mathrm{d}z - \frac{kRa^2}{\sigma p} \int |w|^2\,\mathrm{d}z - \frac{kRa^2 s}{\sigma p} \int w^*(D^2-a^2)\eta\,\mathrm{d}z$$
$$= -Ra^2 \int \theta(D^2 - a^2 - \sigma p^*)\theta^*\,dz - \frac{kRa^2}{\sigma p} \int |w|^2\,\mathrm{d}z$$
$$+ \frac{kRa^2 s}{\sigma p} \int [(D^2-a^2)\eta][(D^2 - a^2 - p^*\sigma/s)\eta^*]\,\mathrm{d}z$$
$$- \frac{kRa^2 s}{\sigma p} \int \sigma p^*\theta^*(D^2-a^2)\eta\,\mathrm{d}z \tag{12}$$

where in the last step we have used (2) and (3). We note from (3) that

$$\int \theta^*(D^2-a^2)\eta = \frac{1}{s} \int \theta^*(D^2-a^2)\theta\,\mathrm{d}z + \frac{\sigma}{s}p \int \theta^*\eta\,\mathrm{d}z. \tag{13}$$

We use the above, together with (5*a*)-(5*c*) in (12) to find

$$\int |D^2w|^2\,\mathrm{d}z + 2a^2 \int |Dw|^2\,\mathrm{d}z + a^4 \int |w|^2\,\mathrm{d}z + p \int |Dw|^2\,\mathrm{d}z + a^2 p \int |w|^2\,\mathrm{d}z$$
$$= Ra^2 \int |D\theta|^2\,\mathrm{d}z + Ra^4 \int |\theta|^2\,\mathrm{d}z + Ra^2\sigma p^* \int |\theta|^2\,\mathrm{d}z - \frac{kRa^2}{\sigma p} \int |w|^2\,\mathrm{d}z$$
$$+ \frac{kRa^2 s}{\sigma p} \int |(D^2-a^2)\eta|^2\,\mathrm{d}z + \frac{kRa^2 p^*}{p} \int |D\eta|^2\,\mathrm{d}z + \frac{kRa^4 p^*}{p} \int |\eta|^2\,\mathrm{d}z$$
$$+ \frac{kRa^2 p^*}{p}\left(\int |D\theta|^2\,\mathrm{d}z + a^2 \int |\theta|^2\,\mathrm{d}z\right) - kRa^2\sigma p^* \int \theta^*\eta\,\mathrm{d}z. \tag{14}$$

With $p=p_1+ip_2$ and $p_1=0$ at the onset point, by equating the imaginary parts of (14) we obtain

$$p_2\int|Dw|^2\,dz+p_2a^2\int|w|^2\,dz$$
$$=-Ra^2\sigma p_2\int|\theta|^2\,dz+\frac{kRa^2p_2}{\sigma|p|^2}\int|w|^2\,dz-\frac{kRa^2sp_2}{\sigma|p|^2}\int|(D^2-a^2)\eta|^2\,dz$$
$$+p_2kRa^2\sigma\,\mathrm{Re}\int\theta^*\eta\,dz. \tag{15}$$

Since $p_2\neq 0$,

$$\int|Dw|^2\,dz+Ra^2\sigma\int|\theta|^2\,dz+a^2\left(1-\frac{kR}{\sigma|p|^2}\right)\int|w|^2\,dz+\frac{kRa^2s}{\sigma|p|^2}\int|(D^2-a^2)\eta|^2\,dz$$
$$=kRa^2\sigma\,\mathrm{Re}\int\theta^*\eta\,dz$$
$$\leqslant\frac{kRa^2\sigma}{2}\left(\int|\theta|^2\,dz+\int|\eta|^2\,dz\right)$$
$$\leqslant\frac{kRa^2\sigma}{2}\int|\eta|^2\left(1+\frac{\frac{3}{2}k}{1-\frac{3}{2}k}\right)dz$$
$$=\frac{kRa^2\sigma}{2}\frac{1}{1-\frac{3}{2}k}\int|\eta|^2\,dz \tag{16}$$

where we have made use of the inequality of (10).

For $k>\frac{2}{3}$, the right-hand side is negative and this can be true only if $kR/\sigma|p|^2\geqslant 1$, since all the other terms on the left-hand side are positive definite. Since $|p|^2=\omega^2$, where ω is the frequency of the oscillatory state at the onset, this establishes the theorem that

$$\omega^2\leqslant kR/\sigma. \tag{17}$$

How close is the inequality? According to [7] for ^{3}He–^{4}He mixtures an accurate numerical analysis of the linear stability equations yields values of $R=1.14\times10^4$ and $\omega=86.5$ for $k=\frac{3}{2}$ and $\sigma=0.7$. The upper limit according to (17) is $\omega_{\mathrm{max}}=156$. For $k\simeq 1$, $\sigma=0.78$, $R\simeq 0.5\times10^4$, leading to $\omega_{\mathrm{max}}\simeq 80$, while the ω obtained from the numerical work is about 40.

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